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ALGORITHMS AND RESULTS OF SOLVING THE INVERSE HEAT-CONDUCTION BOUNDARY PROBLEM IN A TWO-DIMENSIONAL FORMULATION

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The effectiveness of gradient algorithms for solving the inverse problem which are regulated in terms of the number of iterations is investigated.

The need to solve inverse heat-conduction boundary problems (IHBP) most often arises in analyzing the results of thermal experiments when a quantity characterizing the heat transfer at the boundary surface of the body (heat-flux density, heat-transfer coefficient, surface temperature of body) must be determined from the results of temperature measurements at a series of fixed points of a solid.

In view of the incorrectness of the initial formulation of IHBP associated with violation of the stability conditions, regularizing algorithms are used for their solution [1, 2]. Effective algorithms have now been developed for the inverse problem in the one-dimensional formulation. The region of practical use of methods of identifying and diagnosing heat-transfer processes on the basis of solving one-dimensional IHBP is sufficiently large but nevertheless limited. In a series of cases, it is necessary to resort to IHBP in a two-dimensional formulation. However, in terms of algorithm development, such problems have been inadequately studied.

As shown in [3], efficient algorithms for solving two-dimensional IHBP may be obtained by means of iterative regularization using a different approximation of the heat-conduction boundary problem. This approach is developed below for the integral form of the two-dimensional problem with constant thermophysical characteristics. This formulation covers cases of practical application in which the change in properties of the body in the given tempera-

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ture range may be neglected or the specific features of this change are taken into account so as to reduce the problem to this two-dimensional formulation. For example, as noted in [2], the condition of constant thermal diffusivity is satisfied for a series of metals and nonmetals with acceptable accuracy over a relatively broad temperature range. As a result, it is possible to proceed to a linear inverse problem in terms of the model temperature specified by the Kirchhoff transformation. In these cases, it is expedient to use an integral form of IHBP, which is preferable from the viewpoint of the efficiency of computer realization of the algorithms, in particular in terms of the volume of operative memory required and especially the computation time.

The formulation of the IHBP for a body of rectangular cross section is now developed. Assume that the heat-flux vector at each point of space is parallel to the plane XOY and there is no heat transfer in the planes x = 0, x = b, y = 0. It is required to find the heat-flux density q(x, τ) at the boundary y = c on the line y = 0. Thus, the corresponding relations are

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \quad x \in (0,\ b),\ y \in (0,\ c),\ \tau \in (0,\ \tau_m], \tag{1}$$

$$T(x,\ y,\ 0) = \varphi(x,\ y),\ x \in (0,\ b),\ y \in (0,\ c), \tag{2}$$

$$\frac{\partial T(x,\ 0,\ \tau)}{\partial y} = \frac{\partial T(0,\ y,\ \tau)}{\partial x} = \frac{\partial T(b,\ y,\ \tau)}{\partial x} = 0, \quad \tau \in (0,\ \tau_m], \tag{3}$$

$$T(x,\ 0,\ \tau) = f^*(x,\ \tau), \quad \tau \in (0,\ \tau_m]. \tag{4}$$

Here the constant a and the functions $\varphi(x,\ y)$ and f*(x, τ) are known.

The formulation of this inverse problem in terms of the Green's function is G(x, y; x', y'; τ − τ'):

$$\int_0^{\tau} d\tau' \int_0^{b} q(x',\ \tau')\, G(x,\ 0;\ x',\ c;\ \tau - \tau')\, dx' = f(x,\ \tau),\ \tau \in [0,\ \tau_m], \tag{5}$$

where

$$f(x,\ \tau) = \frac{\lambda}{a}\frac{1}{bc}\left(bcf^*(x,\ \tau) - \int_0^{b} dx' \int_0^{c} \varphi(x',\ y')\, G(x,\ 0;\ x',\ y';\ \tau - \xi)\, dy'\right)$$

$$G(x,\ y;\ x',\ y';\ \tau - \xi) = \left(1 + 2\sum_{r=1}^{\infty} \cos\frac{r\pi x}{b}\cos\frac{r\pi x'}{b}\exp\left(-\frac{ar^2\pi^2(\tau - \xi)}{b^2}\right)\right)\times$$

$$\times\left(1 + 2\sum_{r=1}^{\infty} \cos\frac{r\pi y}{c}\cos\frac{r\pi y'}{c}\exp\left(\frac{ar^2\pi^2(\tau - \xi)}{c^2}\right)\right).$$

Equation (5) with an unknown function q(x', τ') is a two-dimensional linear integral equation of Fredholm–Volterra type of the first kind.

It is known that the problem in Eqs. (1)-(4) is unstable; therefore, approximate methods must be used for its solution, allowing the measure of closeness of the approximate solution to the accurate solution to be controlled. In particular, this category includes iterative gradient methods; below, the method of steepest descent and the conjugate-gradient method will be analyzed. As shown in [4, 5], these methods generate regularizing families of operators and, if the approximation process is terminated according to the discrepancy criterion, they become regularizing algorithms.

Introducing the mean-square discrepancy functional

$$J(q^k) = \int_0^{\tau_m} d\tau \int_0^{b} (T^k(x,\ 0,\ \tau) - f(x,\ \tau))^2\, dx, \quad k = 0,\ 1,\ 2,\ \ldots, \tag{6}$$

where $T^k(x,\ 0,\ \tau)$ is the solution of the direct problem corresponding to specifying a heat flux $q^k(x,\ \tau)$ at the boundary y = c, $q^{k+1}(x,\ \tau)$ is found from the iterative formula

$$q^{k+1}(x,\ \tau) = q^k(x,\ \tau) - \beta_k \xi^k(x,\ \tau), \quad k = \overline{0,\ k^*}. \tag{7}$$

Here the direction of descent $\xi^k(x, \tau)$ is determined as a function of the method employed: in the fastest-descent algorithm

$$\xi^k(x,\ \tau) = J'(q^k), \tag{8}$$

in the conjugate-gradient algorithm

$$\xi^k(x,\ \tau) = J'(q^k) + \gamma_k \xi^{k-1}(x,\ \tau),$$
$$\gamma_k = \frac{A_k}{A_{k-1}},\ \gamma_0 = 0, \quad A_k = \int_0^{\tau_m} d\tau \int_0^b (J'(q^k))^2\, dx, \tag{9}$$

where $J'(q^k)$ is the discrepancy gradient at the k-th iteration.

The coefficient β_k in Eq. (7) is chosen from the condition

$$\beta_k : \min_{\beta \geqslant 0} J(q^k - \beta \xi^k). \tag{10}$$

It is simple to show that

$$\beta_k = \frac{\int_0^{\tau_m} d\tau \int_0^b (T^k(x,\ 0,\ \tau) - f(x,\ \tau))\, \Delta T_k(x,\ 0,\ \tau)\, dx}{\int_0^{\tau_m} d\tau \int_0^b \Delta T_k^2(x,\ 0,\ \tau)\, dx}. \tag{10'}$$

Here $\Delta T_k(x, 0, \tau)$ is the temperature increment corresponding to a variation in heat-flux density of $\xi^k(x, \tau)$. This quantity is found from the solution of the following problem:

$$\frac{\partial \Delta T}{\partial \tau} = a\left(\frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{\partial y^2}\right), \quad x \in (0,\ b),\ y \in (0,\ c),\ \tau \in (0,\ \tau_m],$$
$$\Delta T(x,\ y,\ 0) = 0,$$
$$\frac{\partial \Delta T(b,\ y,\ \tau)}{\partial x} = \frac{\partial \Delta T(0,\ y,\ \tau)}{\partial x} = \frac{\partial \Delta T(x,\ 0,\ \tau)}{\partial y} = 0, \tag{11}$$
$$-\lambda \frac{\partial \Delta T(x,\ c,\ \tau)}{\partial y} = \xi^k(x,\ \tau).$$

The number of the last iteration k* is established from the discrepancy criterion

$$k^* : J(q^k) \simeq \delta^2, \tag{12}$$

where δ is the total error, including the error in the temperature information δ_T and the error in approximating the heat-conduction boundary problem δ_α ($\delta^2 = \delta^2_\alpha + \delta^2_T$)

At iteration number k, the temperature $T^k(x, 0, \tau)$ will be determined from the solution of the direct heat-conduction problem from the specified approximation to the true function, i.e., from the solution of Eq. (1) with the boundary conditions in Eqs. (2) and (3) and the condition

$$-\lambda \frac{\partial T(x,\ c,\ \tau)}{\partial y} = q^k(x,\ \tau), \quad \tau \in [0,\ \tau_m].$$

The gradient of the functional in Eq. (6) is found in terms of the conjugate variable $J'(q) = -\psi(x, c, \tau)$. The function $\psi(x, c, \tau)$ satisfies the boundary problem

$$\frac{\partial \psi}{\partial \tau} = -a\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right), \quad x \in (0,\ b), \quad y \in (0,\ c), \quad \tau \in (0,\ \tau_m], \tag{13}$$

$$\psi(x,\ y,\ \tau_m) = 0,$$
$$\frac{\partial\psi(x,\ c,\ \tau)}{\partial y} = \frac{\partial\psi(b,\ y,\ \tau)}{\partial x} = \frac{\partial\psi(0,\ y,\ \tau)}{\partial x} = 0, \tag{13}$$
$$\frac{\partial\psi(x,\ 0,\ \tau)}{\partial y} = 2(T(x,\ 0,\ \tau) - f(x,\ \tau)).$$

The solution of the direct problem T(x, 0, τ), the conjugate problem ψ(x, c, τ), and the problem in terms of the increments ΔT(x, 0, τ) required for the organization of the iterative process are found by means of the Green's function.* The universal form of this procedure is

$$z(x,\ \tau) = \int_0^{\tau} d\tau' \int_0^{b} F(x',\ \tau')\, G(x,\ y_*;\ x',\ y'_*;\ \tau - \tau')\, dx'. \tag{14}$$

For the direct problem

$$z(x,\ \tau) = T(x,\ 0,\ \tau), \quad F(x',\ \tau') = q(x',\ \tau'), \quad y_* = 0, \quad y'_* = c,$$

for the conjugate problem

$$z(x,\ \tau) = \psi(x,\ \tau), \quad F(x',\ \tau') = -2\lambda(T(x',\ 0,\ \tau') - f(x',\ \tau')), \quad y_* = c, \quad y'_* = 0,$$

and for the problem in terms of increments

$$z(x,\ \tau) = \Delta T(x,\ 0,\ \tau), \quad F(x',\ \tau') = \xi(x',\ \tau'), \quad y_* = 0, \quad y'_* = c.$$

In calculating the functions T(x, 0, τ), ξ(x, τ) and ΔT(x, 0, τ), an approximate–analytic method is used [2]: in the region of solution $D: \{x \in [0, b],\ y \in [0, c],\ \tau \in [0, \tau_m]\}$ a grid with points $x_l = l\Delta x$, $l = \overline{0, L}$, $\tau_j = j\Delta\tau$, $j = \overline{0, m}$, $y_k = k\Delta y$, $k = \overline{0, K}$ is introduced, and Eq. (14) transforms to an approximating system of linear algebraic equations. This transformation is made under the condition of analytic integration of the kernel of the right-hand side of Eq. (14) in the grid cell.

Equation (5) is written in the form

$$Aq = v, \quad q \in Q, \quad v \in V, \tag{15}$$

where Q and V are normalized spaces and A : Q → V is a linear integral operator determining the Green's function. After approximating the problem in Eq. (15), the operator A will be replaced by the matrix A_Δ and the continuous functions q and v by the vectors **q** and **v**, which are obtained as a result of stepwise approximation over time and space in the form

$$\mathbf{v} = [\tilde{v}_{11},\ \tilde{v}_{21},\ \ldots,\ \tilde{v}_{L1};\ \tilde{v}_{12},\ \tilde{v}_{22},\ \ldots,\ \tilde{v}_{L2};\ \ldots;\ \tilde{v}_{1m},\ \tilde{v}_{2m},\ \ldots,\ \tilde{v}_{Lm}]^{\mathrm{T}},$$

where

$$\tilde{v}_{lj} = \frac{\hat{v}_{lj-1} + \hat{v}_{lj}}{2}; \quad \hat{v}_{lj} = \frac{v_{l-1}(\tau_j) + v_l(\tau_j)}{2}; \quad v_l(\tau_j) = v(x_l,\ \tau_j).$$

With the appropriate choice of parameters, the difference grid will be a sufficiently accurate approximation of the integral equation, which allows the corresponding error to be disregarded in constructing the algorithm for solution of the inverse problem. As an illustration of this, Table 1 gives the temperature values at an arbitrarily chosen space–time point (x = 0.005 m, τ = 0.25 sec) with various difference-grid parameters (L is the number of steps along the x axis, and M the number along the τ axis: τ_m = 1.0 sec, b = 0.01 m, c = 0.02 m, ΔFo = 0.014, $q_{max} = 0.92 \cdot 10^{+6}$ W/m²).

At the same time, the right-hand side of Eq. (15) may be known with considerable errors due to imprecise measurements of the temperature and errors in deciphering the experimental information. Therefore, it is assumed below that, instead of the element v, the element $v_\delta = v + \Delta v$ is known, where Δv is the noise on the right-hand side of Eq. (15), with the norm $\|\Delta v\|_{L_2} = \delta_1$. Thus, the error in solving Eq. (15) under the assumption that the error in the approximation may be neglected will be equal to the error on the right-hand side δ_T.

*For this purpose, the conjugate problem in Eq. (13) is reduced to a formulation with "inverse" time by the variable substitution $t = \tau_m - \tau$.

TABLE 1. Temperature Values at an Arbitrarily Chosen Point with Various Difference-Grid Parameters

M	L				
	2	4	6	8	12
2	0,1726	0,1728	0,1729	0,1729	0,1729
4	0,9989	0,1000	0,1000	0,1000	0,1000
8	0,0848	0,0849	0,0849	0,0849	0,0849
12	0,0819	0,0820	0,0821	0,0821	0 0821
16	0,0809	0,0809	0,0810	0,0811	0,0311
20	0,0804	0,0805	0,0806	0,0806	0,0806

In calculating the elements of matrix A_Δ, the sum of infinite series must be determined; see Eq. (5). Truncating the series leads to the appearance of additional errors, as a result of which the matrix $A_{\Delta H} = A_\Delta + H$ will be calculated instead of the matrix A_Δ, where H is the error matrix. Considerations of machine-time economy demand the summation of only as many terms in the series as is necessary for the solution of the problem with no worse than the specified accuracy.

The error appearing on account of truncation of the infinite series must be negligibly small in comparison with the error on the right-hand side

$$\| Hq \|_{E_n} \ll \| \Delta v \|_{E_n}, \tag{16}$$

where $\| \cdot \|_{E_n}$ is the norm of the Euclidean space of n-dimensional real vectors; $n = L \cdot m$. It is necessary to estimate $\| Hq \|_{E_n}$.

It is assumed that all the elements of matrix H are equal to some positive number h, and that the series which are calculated in determining the elements of matrix A_Δ are variable in sign and absolutely convergent (or may be reduced to this form by means of transformations). In that case, it will be simple to establish the majorant error in calculating the series: it will be equal to the last term being summed in the series and, of course, this last term must be positive.

The norm $\| \cdot \|_\infty$ is used, with the following definition. If $x = \{x_k\}$, $k = \overline{1, n}$, then $\| x \|_\infty = \max\limits_{1 \leqslant k \leqslant n} |x_k|$. It is simple to show that $\| Hq \|_\infty \leqslant mLq_{max}h$, where q_{max} is the maximum value of the heat-flux density. Thus, Eq. (16) will be satisfied if $mLq_{max}h \ll \Delta f_{max}$, where Δf_{max} is the maximum error in measuring the temperature. Hence, if the error appearing as a result of the truncation of the infinite series is to be negligibly small in comparison with the error on the right-hand side of Eq. (15), the condition $h \ll \Delta f_{max}/mLq_{max}$ must be satisfied.

In accordance with the above algorithm for IHBP solution, a program has been written for use in computational experiments on an EC-1033 computer. In each experiment, the temperature $T(x, 0, \tau)$ is determined for some heat-flux distribution at the boundary $y = c$ (direct problem) and hence the heat flux acting at the boundary $y = c$ is calculated (inverse problem).

The criterion adopted for the recovery of the initial function $q(x, \tau)$ is the normalized functional

$$\overline{J}(q) = J(q) / \int_0^{\tau_m} d\tau \int_0^b (f(x, \tau))^2 dx.$$

The initial approximation adopted is $q^0(x, \tau) = 0$.

The results of recovering the function $q(x, \tau)$ using the method of fastest descent with accurate input data are shown in Fig. 1, I. The quality of the resulting approximation is poor, although sufficiently many iterations were taken ($k = 75$). The time per iteration is $t_i = 12$ sec.

In Fig. 1, II, the results of solving the IHBP by the conjugate-gradient method with accurate input data are shown. It is evident that the recovery of $q(x, \tau)$ is better over all. The computation time is slightly greater: $t_i = 12.6$ sec in this case. Also evident from Fig. 1, II is the significant deterioration in accuracy of the mean-square approximation ob-

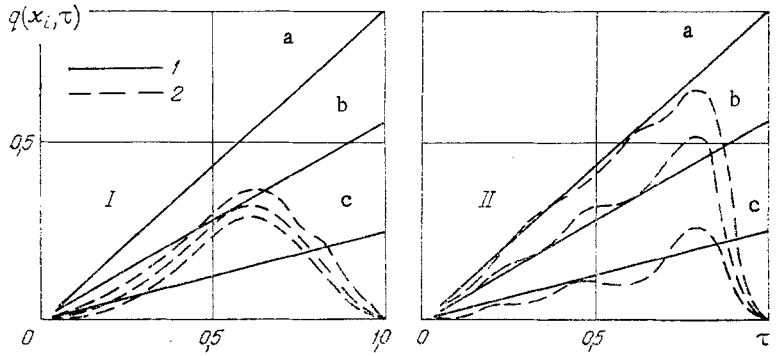

Fig. 1. Approximation on average to the heat-flux density by the method of fastest descent (I) and the conjugate-gradient method (II) (c = 0.02 m, b = 0.01 m, ΔFo = 0.014, q (W/m^2)·10^{-6}; τ (sec)): 1) actual solution; 2) 75th (I) and 50th (II) iteration; a) $q(x_1, \tau)$; b) $q(x_3, \tau)$; c) $q(x_5, \tau)$.

tained in the vicinity of the end of the time interval ($\tau = \tau_m$). This may readily be explained. In fact, the gradients of the functional in the k-th and (k − 1)-th iterations — $J'(q^k)$ and $J'(q^{k-1})$ — appear in the expression for $\xi^k(x, \tau)$ in Eq. (9). Their accurate values at $\tau = \tau_m$ are zero, which follows from the formulation of the conjugate problem in Eq. (13). Thus, with accurate calculation of the gradient, the solution $q(x, \tau)$ at point $\tau = \tau_m$ in all the iterations must be equal to the initial approximation at this point.

The deterioration in the recovery of the function in the vicinity of the end of the time interval may be eliminated if account is taken of the smoothness of the given function in constructing the iterative algorithm [2].

Suppose that the function $q(x, \tau)$ has a mixed derivatives $q_{x\tau}(x, \tau) = \partial^2 q(x, \tau)/\partial x \partial \tau \in L_2$ and the target functional in Eq. (6) is specified. Consider the iterative process of determining this derivative by the conjugate-gradient method, where $q^0_{x\tau}(x, \tau)$ is the specified initial approximation

$$q^{k+1}_{x\tau}(x, \tau) = q^k_{x\tau}(x, \tau) - \beta_k S^k_{x\tau}, \quad k = 0, 1, 2, \ldots, \tag{17}$$

where β_k is the step in the direction toward the minimum of $J(q)$, and

$$S^k_{x\tau} = J'(q^k_{x\tau}) + \gamma_k S^{k-1}_{x\tau}; \quad \gamma_k = \frac{A_k}{A_{k-1}}; \quad \gamma_0 = 0; \quad A_k = \int_0^{\tau_m} d\tau \int_0^b (J'(q^k_{x\tau}))^2 \, dx. \tag{18}$$

Integrating Eq. (17) with respect to x and τ gives

$$q^{k+1}(x, \tau) = q^k(x, \tau) + \Delta q_k(x, 0) + \Delta q_k(b, \tau) - \Delta q_k(b, 0) + \beta_k \int_0^\tau d\tau' \int_x^b S^k_{x\tau}(x', \tau') \, dx', \tag{19}$$

where $\Delta q_k(x, 0)$, $\Delta q_k(b, \tau)$, and $\Delta q_k(b, 0)$ are increments in the function $q(x, \tau)$ at the (k + 1)-th iteration, respectively, at the boundaries of the region of solution $\tau = 0$, $x = b$, and at the point $x = b$, $\tau = 0$. The values of $\Delta q_k(x, 0)$, $\Delta q_k(b, \tau)$, and $\Delta q_k(b, 0)$ are determined as follows

$$\Delta q_k(x, 0) = \beta_{xk} \int_x^b S^k_x(x') \, dx' + \Delta q_k(b, 0),$$

$$\Delta q_k(b, \tau) = -\beta_{\tau k} \int_0^\tau S^k_\tau(\tau') \, d\tau' + \Delta q_k(b, 0),$$

$$\Delta q_k(b, 0) = -\beta_{bk} S^k.$$

where $S^k{}_x(x)$, $S^k{}_\tau(\tau)$, and S^k are defined analogously to Eq. (18); the formulas for the gradients of the functional take the following form

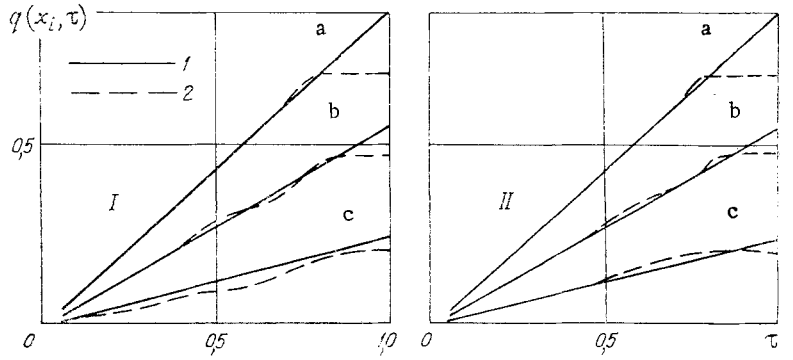

Fig. 2. Recovery of heat-flux density by the conjugate-gradient method, taking account of the smoothness of the solution for the cases when all the variations are determined (I) and when the variations $\Delta q_k(x, 0)$, $\Delta q_k(b, \tau)$, and $\Delta q_k(b, 0)$ are not determined (II): 1) the actual solution; 2) 30th (I) and 20th (II) iteration. Other notation as in Fig. 1.

$$J'(q(b, 0)) = w(b, 0),\ J'(q_\tau(b, \tau)) = w(b, \tau),\ \tau \in [0, \tau_m],$$
$$J'(q_x(x, 0)) = -w(x, 0),\ x \in [0, b],$$
$$J'(q_{x\tau}(x, \tau)) = -w(x, \tau),\ x \in [0, b],\ \tau \in [0, \tau_m],$$

where

$$w(x, \tau) = \int_\tau^{\tau_m} d\tau' \int_0^x J'(q(x', \tau'))\, dx',\ x \in [0, b];\ \tau \in [0, \tau_m],$$

and, as already noted, $J'(q(x, \tau))$ is determined from the solution of the conjugate problem.

The step in the descent to the minimum of the functionals β_k, β_{xk}, $\beta_{\tau k}$, and β_{bk} is found from the condition [6]:

$$J(q^{k+1}) = \min_{\substack{\beta_k,\ \beta_{xk} \\ \beta_{\tau k},\ \beta_{bk}}} J(q^k + \beta_{xk}\int_x^b S_x^k\, dx' - \beta_{\tau k}\int_0^\tau S_\tau^k\, d\tau' - \beta_{bk} S^k + \beta_k \int_0^\tau d\tau' \int_x^b S_{x\tau}^k\, dx'). \tag{20}$$

The necessary condition for an extremum of the multidimensional function leads to the following system of algebraic equations

$$\begin{aligned}
A_{11}\beta_k + A_{12}\beta_{xk} - A_{13}\beta_{\tau k} - A_{14}\beta_{bk} &= B_1,\\
A_{21}\beta_k + A_{22}\beta_{xk} - A_{23}\beta_{\tau k} - A_{24}b_{bk} &= B_2,\\
A_{31}\beta_k + A_{32}\beta_{xk} - A_{33}\beta_{\tau k} - A_{34}\beta_{bk} &= B_3,\\
A_{41}\beta_k + A_{42}\beta_{xk} - A_{43}\beta_{\tau k} - A_{44}\beta_{bk} &= B_4.
\end{aligned} \tag{21}$$

Here

$$A_{ij} = \int_0^{\tau_m} d\tau \int_0^b \Delta T(N_i)\, \Delta T(N_j)\, dx;\quad N_1 = \int_0^\tau d\tau' \int_x^b S_{x\tau}^k\, dx';$$
$$N_2 = \int_x^b S_x^k\, dx';\ N_3 = \int_0^\tau S_\tau^k\, d\tau';\ N_4 = S^k,$$

and $\Delta T(N_i)$ is the temperature increment due to the corresponding variation N_i, $i = \overline{1, 4}$.

The results of solving IHBP by the given method are shown in Fig. 2, I. Note the better quality of the recovery of the given function over the whole region. The time to perform a single iteration in this case is approximately doubled ($t_i \approx 26$ sec); however, the number of iterations required is reduced, and thus the total computer time required is unchanged.

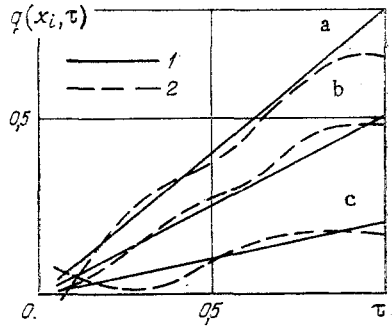

Fig. 3. Recovery of heat-flux density by the conjugate-gradient method, taking account of the smoothness of the solution, for perturbed data: 1) actual solution; 2) results of calculation from the temperature values with random errors distributed according to a normal law with a spread $3\sigma = 0.01T_{max}$. Other notation as in Fig. 1.

The computation time when using the gradient method of smooth regularization may be reduced if the behavior of the function $q(x, \tau)$ is known at the limits of the region of solution $\tau = 0$ and $x = b$. In this case, there is no need to determine the increment in the function at the limit $\Delta q_k(x, 0)$, $\Delta q_k(b, \tau)$ and at the point $\Delta q_k(b, 0)$ — see Eq. (19) — and hence the computation time is considerably reduced. In Fig. 2, II, the results of recovering the function $q(x, \tau)$ are shown, under the assumption that $q(x, 0) = q(b, \tau) = 0$. However, this approach is not very suitable if the behavior of the function $q(x, \tau)$ at the limits of the region is hard to analyze.

The influence of perturbations of the initial data on the solution of the inverse problem has also been analyzed, by means of computational experiments in which the solution of the direct problem is perturbed by a random-number sensor according to a normal law of random-error distribution. Termination of the iterative process is based on the condition in Eq. (12), where it is assumed that

$$\delta_a = 0, \ \delta_\tau^2 = \int_0^{\tau_m} d\tau \int_0^b \sigma^2 dx,$$

$\sigma^2(x, \tau)$ is the dispersion of the specified temperature function. The results of solving one example are shown in Fig. 3.

The quality of recovery is better; the results of these computational experiments permit the conclusion that the given algorithm is suitable for the solution of IHBP using perturbed input data.

NOTATION

T, temperature; x, y, spatial coordinates; τ, time; λ, thermal conductivity; a, thermal diffusivity; q, heat-flux density; $\Delta Fo = a\Delta\tau/c^2$, step in the Fourier number; k, iteration number; ξ^k, direction of descent in k-th iteration; J(q), mean-square discrepancy functional; E_n, Euclidean space of n-dimensional real vectors; A, linear integral operator; A_Δ, matrix of approximating operator; δ_a, δ_T, errors due to the approximation of the problem and the inaccuracy of the temperature information; σ^2, dispersion of the specified function of the temperature; t_i, time for one computer iteration.

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